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Enhancement of stability in randomly switching potential with metastable state

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Abstract. The overdamped motion of a Brownian particle in randomly switching piece-wise metastable linear potential shows noise enhanced stability (NES): the noise stabilizes the metastable system and the system remains in this state for a longer time than in the absence of white noise. The mean first passage time (MFPT) has a maximum at a finite value of white noise intensity. The analytical expression of MFPT in terms of the white noise intensity, the parameters of the potential barrier, and of the dichotomous noise is derived. The conditions for the NES phenomenon and the parameter region where the effect can be observed are obtained. The mean first passage time behaviors as a function of the mean flipping rate of the potential for unstable and metastable initial configurations are also analyzed. We observe the resonant activation phenomenon for initial metastable configuration of the potential profile.

PACS. 05.40.-a Fluctuation phenomena, random processes, noise, and Brownian motion – 02.50.-r Probability theory, stochastic processes, and statistics – 05.10.Gg Stochastic analysis methods (Fokker-Planck, Langevin, etc.)

1 Introduction

The thermally activated escape from a metastable state in fluctuating potential is of great importance to many natural systems, ranging from physical and chemical systems to biological complex systems. A particular challenging direction are systems far from thermal equilibrium, either due to non-thermal noise or external deterministic periodical forces [1–11]. A typical problem is the enhancement of stability of metastable and unstable states due to the external noise [7–11]. The noise-enhanced stability (NES) phenomenon was observed experimentally and numerically in various physical systems (see Refs. [5–14] and, as a recent review, reference [15]). The investigated systems were subjected to the action of two forces: additive white noise and regular force. The regular force was fixed or periodical in time, so that metastable state appeared for a half of period and unstable state appeared for an other half. The noise enhanced stability effect implies that, under the action of additive noise, a system remains in the metastable state for a longer time then in the deterministic case, and the escape time has a maximum as a function of noise intensity. We can lengthen or shorten the mean lifetime of the metastable state of our physical system, by acting on the white noise intensity. The noise-induced stabilization of one-dimensional maps [16,17], the noise-induced stability in fluctuating bistable potentials [18], the noise induced slowing down in a periodical potential [19–21], the noise induced order in one-dimensional map of the Belousov-Zhabotinsky reaction [22,23], and the transient properties of a bistable kinetic system driven by two correlated noises [24], are akin to the NES phenomenon. Moreover, this effect is at the basis of resonant trapping [25]. Even though the previous theoretical papers analyzed NES phenomenon in systems with fixed or periodically driving metastable and unstable states, the model of randomly switching metastable state is more realistic in many cases, e.g. when we describe the generation process of the carrier traps in semiconductors. Despite its experimental importance, the theory of fluctuating barrier crossing is not well developed for arbitrary noise intensity. In the present paper we obtain and study analytically the NES effect in a system described by a potential, which randomly switches between metastable and unstable configurations. We define the lifetime of metastable state as the mean first passage time (MFPT) and obtain the conditions when it can grow with white noise intensity. The mean first passage time behaviour as a function of the mean flipping rate of the potential for unstable and metastable initial configurations is also analyzed. We observe the resonant activation phenomenon for initial metastable configuration of the potential profile,

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and monotonic behaviour for initial unstable configuration. We can describe therefore, with the same theoretical approach, two noise-induced phenomena [7,10,15,26–28].

2 The model

We consider one-dimensional overdamped Brownian motion in a randomly switching potential profile $U(x)$

$$\frac{dx}{dt} = f(x) - a\eta(t) + \xi(t), \tag{1}$$

where $\xi(t)$ is the white Gaussian noise with zero mean and $\langle\xi(t)\xi(t+\tau)\rangle = 2D\delta(\tau)$, $\eta(t)$ is a Markovian dichotomous process, which takes the values ± 1 with mean flipping rate ν, and $f(x) = -dU(x)/dx$. We investigate the mean first passage time with the reflecting boundary at the point $x = 0$ and the absorbing boundary at the point $x = b$ $(b > 0)$. The calculation technique for the MFPT of non-Markovian process $x(t)$ without white noise ($\xi(t) = 0$) has been originally developed in reference [29] and, then, generalized by various authors (see, for example Refs. [30–32]). The exact equations for mean first passage times, which take into account both the dichotomous and the white noise terms in equation (1), were derived in reference [33], where the authors solve the delicate problem to construct correct conditions at the reflecting boundary, since all previous works dealt with absorbing boundaries only. Nevertheless the cited boundary conditions (3.7a) and (3.7b) in reference [33] are only valid for the special unlikely case of immediate reflection. For our purposes we use the same conditions at the reflecting boundary as in reference [34]. Thus from the backward Fokker-Planck equation we obtain the coupled differential equations for the MFPTs in our system (1) (see Ref. [33] and Appendix A)

$$\begin{aligned} DT_+'' + [f(x) - a]T_+' + \nu(T_- - T_+) &= -1, \\ DT_-'' + [f(x) + a]T_-' + \nu(T_+ - T_-) &= -1. \end{aligned} \tag{2}$$

Here $T_+(x)$ and $T_-(x)$ are respectively the mean first passage times of the boundary $x = b$ for initial values $\eta(0) = +1$ and $\eta(0) = -1$, with the Brownian particle starting from the initial point x $(0 < x < b)$. Our conditions at the reflecting boundary $x = 0$ and the absorbing boundary $x = b$ are (see Appendix B)

$$T_\pm'(0) = 0, \qquad T_\pm(b) = 0. \tag{3}$$

Let us introduce for convenience two new auxiliary functions

$$T = \frac{T_- + T_+}{2}, \qquad \Theta = \frac{T_- - T_+}{2}, \tag{4}$$

and rewrite equations (2) in more simple form

$$\begin{aligned} DT'' + f(x)T' + a\Theta' &= -1, \\ D\Theta'' + f(x)\Theta' + aT' - 2\nu\Theta &= 0. \end{aligned} \tag{5}$$

The boundary conditions for the functions $T(x)$ and $\Theta(x)$ follow from equations (3) and (4)

$$T'(0) = \Theta'(0) = 0, \qquad T(b) = \Theta(b) = 0. \tag{6}$$

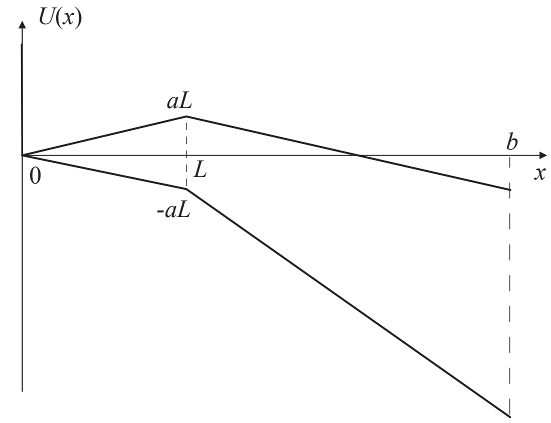

Fig. 1. Two configurations of fluctuating potential $U(x)$, corresponding to the metastable and unstable states.

Equations (5) with the boundary conditions (6) were solved in reference [34] for a model with constant force

$$f(x) = \text{const.}, \qquad 0 < x < b. \tag{7}$$

In the present paper we consider more general model with step-wise force (or piece-wise potential $U(x)$) as

$$f(x) = -\frac{dU(x)}{dx} = k \cdot \theta(x - L), \qquad 0 < L < b, \tag{8}$$

where $\theta(x)$ is the step function and k is a constant. As we can see from Figure 1, if $0 < a < k$ we have a metastable state for $\eta(t) = +1$ and an unstable state for $\eta(t) = -1$.

In particular cases when $L \to 0$ or $L \to b$ the force (8) coincides with (7). After removing $T(x)$ from equation (5) we obtain third-order linear differential equation for the variable $\Theta(x)$

$$\Theta''' + \frac{2f(x)}{D}\Theta'' + F(x)\Theta' - \frac{2\nu f(x)}{D^2}\Theta = \frac{a}{D^2}, \tag{9}$$

where

$$F(x) = \frac{f^2(x)}{D^2} + \frac{f'(x)}{D} - \gamma^2, \quad \gamma = \sqrt{\frac{a^2}{D^2} + \frac{2\nu}{D}}. \tag{10}$$

We will consider the mean first passage times $T_\pm(0)$ with the starting position of Brownian particle at $x = 0$. In the absence of switchings and white noise the dynamical escape time $T_+(0)$ for the initial metastable state is equal to $+\infty$ and the dynamical escape time $T_-(0)$ for the initial unstable state equals

$$T_-(0) = \frac{L}{a} + \frac{b - L}{k + a}. \tag{11}$$

We solve equations (5) and (9) for regions $0 < x < L$ and $L < x < b$ separately. First of all we find the solutions in the interval (L, b)

$$\begin{aligned} \Theta(x) &= \sum_{i=1}^{3} c_i e^{\lambda_i(x-L)} - \frac{a}{2\nu k}, \\ T(x) &= c_4 - \frac{x - L}{k} - a\sum_{i=1}^{3} \frac{c_i\left[e^{\lambda_i(x-L)} - 1\right]}{k + \lambda_i D}, \end{aligned} \tag{12}$$

where c_i $(i = 1 \div 4)$ are unknown constants and λ_1, λ_2, λ_3 are the roots of the following cubic equation

$$\lambda\left(\lambda D + k\right)^2 - \Gamma^2\lambda - 2\nu k = 0, \qquad (13)$$

where $\Gamma = \gamma D$. Using graphical representation it can be easily shown that algebraic equation (13) has three real roots: one positive and two negative ones.

Taking into account the conditions (6) at the reflecting boundary $x = 0$ we obtain for region $(0, L)$

$$\begin{aligned}\Theta(x) &= c_5\left(1 + \frac{2\nu D}{a^2}\cosh\gamma x\right) + \frac{a}{\Gamma^3}\left(D\sinh\gamma x - \Gamma x\right),\\ T(x) &= -\frac{\nu x^2}{\Gamma^2} - \frac{a^2 D}{\Gamma^4}\left(\cosh\gamma x - 1\right)\\ &\quad - c_5\frac{2\nu}{\Gamma a}\left(D\sinh\gamma x - \Gamma x\right) + c_6. \end{aligned} \qquad (14)$$

From equations (4) and (14) we get finally

$$T_{\pm}(0) = T(0) \mp \Theta(0) = c_6 \mp c_5\frac{\Gamma^2}{a^2}. \qquad (15)$$

The six unknown constants c_i $(i = 1 \div 6)$ can be determined from the conditions (6) at the absorbing boundary $x = b$ and the continuity conditions of the functions $\Theta(x)$, $\Theta'(x)$, $T(x)$, $T'(x)$ at the point $x = L$. Thus, we obtain the following system of algebraic linear equations for the constants c_i

$$\begin{aligned}&\sum_{i=1}^{3} c_i e^{\mu_i} = \frac{a}{2\nu k},\\ &\sum_{i=1}^{3}\frac{c_i a\left(e^{\mu_i} - 1\right)}{k + \lambda_i D} = c_4 - \frac{b-L}{k},\\ &\sum_{i=1}^{3} c_i = c_5\left(1 + \frac{2\nu D}{a^2}\cosh\gamma L\right)\\ &+\frac{a}{\Gamma^3}\left(D\sinh\gamma L - \Gamma L\right) + \frac{a}{2\nu k},\\ &\sum_{i=1}^{3} c_i\lambda_i = c_5\frac{2\nu\Gamma}{a^2}\sinh\gamma L + \frac{a}{\Gamma^2}\left(\cosh\gamma L - 1\right),\\ &\sum_{i=1}^{3}\frac{c_i\lambda_i a}{k + \lambda_i D} = \frac{2\nu L}{\Gamma^2} + \frac{a^2}{\Gamma^3}\sinh\gamma L\\ &+\frac{2\nu}{a}c_5\left(\cosh\gamma L - 1\right) - \frac{1}{k},\\ &c_4 = c_6 - \frac{\nu L^2}{\Gamma^2} - \frac{a^2 D}{\Gamma^4}\left(\cosh\gamma L - 1\right)\\ &-c_5\frac{2\nu}{\Gamma a}\left(D\sinh\gamma L - \Gamma L\right), \end{aligned} \qquad (16)$$

where $\mu_i = \lambda_i(b - L)$. The equation (15) with algebraic system (16) is the exact solution of equations (2) for the force (8). In the limits $L \to 0$ or $L \to b$, the expression (15) coincides with that obtained in reference [34] for the force (7).

3 NES phenomenon conditions

Our aim is to investigate the noise enhanced stability effect. In other words we are interested in the situation, when the escape time grows with noise. We expect to find this growing in the limit of small intensity of the white noise. Hereafter we write the mean first passage times $T_{\pm}(0)$ as $T_{\pm}(\nu, D)$ to point out the dependence on the parameters ν and D of two noise sources. On cumbersome rearrangements (see Appendix C), from equations (15) and (16) we obtain in the limit $D \to 0$

$$T_{-}(\nu, D) \simeq T_{-}(\nu, 0) + \frac{D}{a^2}G(q,\omega,s) + o(D), \qquad (17)$$

where

$$T_{-}(\nu, 0) = \frac{2L}{a} + \frac{\nu L^2}{a^2} + \frac{b-L}{k} - \frac{q(1-q)}{2\nu}\left(1 - e^{-s}\right) \qquad (18)$$

is the MFPT in the absence of white noise, and

$$\begin{aligned}G(q,\omega,s) &= \frac{q^3\left(1+q^2\right)se^{-s}}{(1+q)\left(1-q^2\right)} - \frac{5 + q - 5q^2 - 5q^3}{2(1+q)\left(1-q^2\right)}\\ &+ \frac{q\left(1 + q - 5q^2 - 3q^3 - 2q^4\right)}{2(1+q)\left(1-q^2\right)}\left(1 - e^{-s}\right)\\ &+ \frac{2\omega\left(3q^2 + q - 3\right)}{q\left(1-q^2\right)} - \frac{2\omega^2}{q^2}. \end{aligned} \qquad (19)$$

Here q, ω and s are the dimensionless parameters

$$q = \frac{a}{k}, \quad \omega = \frac{\nu L}{k}, \quad s = \frac{2\omega}{1-q^2}\left(\frac{b}{L} - 1\right). \qquad (20)$$

As one would expect, the MFPT (18) coincides with the nonlinear relaxation time (NLRT) for the same system (see Eq. (32) in Ref. [36]) because in the absence of diffusion $(D = 0)$, after crossing the point $x = b$ Brownian particles cannot come back in the interval $(0, b)$ again.

The sign of the term $G(q,\omega,s)$ in equation (19) allows to obtain the conditions to observe the NES effect in the system investigated. We can write these conditions as

$$G(q,\omega,s) > 0. \qquad (21)$$

Let us analyse the structure of NES phenomenon region on the plane (q,ω). In the case of very slow switchings $\nu \to 0$ $(\omega \to 0,\ s \to 0)$ equation (21) takes the form

$$\omega < \frac{q(1-q)\left(5q^3 + 5q^2 - q - 5\right)}{2\left[6 - 2q - q^2(b/L + 11) + 2q^3 + 3q^4(b/L + 1)\right]}, \quad q > 0,8024. \qquad (22)$$

In the case of $q \simeq 1$ we obtain from equation (21)

$$\omega < \frac{x_0(1-q)}{b/L - 1}, \quad \frac{1}{2} + \frac{5}{2}(1-q) < \omega < \frac{1}{2(1-q)}, \qquad (23)$$

where x_0 is the positive root of algebraic equation

$$e^x = x + 2$$

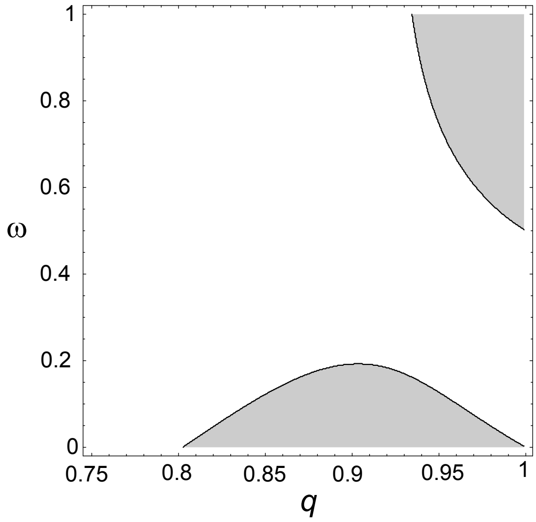

Fig. 2. The shaded areas are the regions of the plane (q, ω) where the NES effect takes place. The parameter is $b/L = 1.5$.

approximately equals $x_0 \simeq 1.1463$. The two shaded areas on the plane (q, ω), where the NES phenomenon takes place, are shown in Figure 2. Both noise enhanced stability areas connect at the parameter $b/L < 1.2655$.

Now we may conclude that the NES effect occurs at the values of q near 1, i.e. at very small steepness $k - a = k(1 - q)$ of the reverse potential barrier for the metastable state (see Fig. 1). Only in such a situation, Brownian particles can move back into potential well in the interval $(0, L)$ from the region $L < x < b$, with a low noise intensity. In reference [10] we obtained the parameter region of NES effect for periodical driven metastable state. This region is larger than that obtained here for randomly switching potential. In our opinion, this narrowing of NES phenomenon area is attributable to the uncertainty of the first switching time. Moreover, because of the exponential probability distribution of the time interval between jumps of the dichotomous noise, which randomly switches our metastable state, the average escape time has a large variance and thus the analogy between the two cases falls.

It must be emphasized that noise enhanced stability regions of the mean first passage time are inside that obtained for the nonlinear relaxation time with the same parameter b/L (compare Eq. (19) with Eq. (31) in Ref. [36]) because the nonlinear relaxation time takes into account a repeated reentries of Brownian particles in the interval $(0, b)$. The NES effect disappears when the absorbing boundary is placed at the point $x = L$. In fact by putting $b \to L$, $k \to +\infty$ $(q \to 0,\ s \to 0)$ in equation (17) we obtain

$$T_-(0) \simeq \frac{2L}{a} + \frac{\nu L^2}{a^2} - \frac{D}{a^2}\left(\frac{5}{2} + \frac{6\nu L}{a} + \frac{2\nu^2 L^2}{a^2}\right). \quad (24)$$

The behaviors of the MFPTs $T_-(\nu, D)$ for the initial unstable configuration, normalized to the value obtained in the absence of white noise $T_-(\nu, 0)$, vs the white noise intensity are shown in Figures 3 and 4 for the two NES areas represented in Figure 2. As we can see the effect

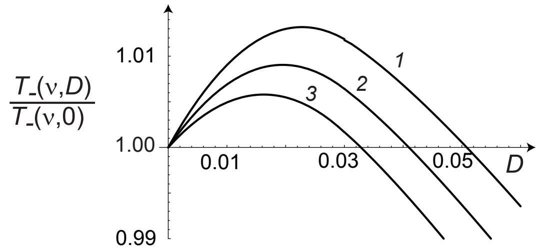

Fig. 3. The normalized MFPT vs the white noise intensity D, for three values of the dimensionless mean flipping rate $\omega = (\nu L)/k$: 0.01 (curve 1), 0.04 (curve 2), 0.07 (curve 3) (lower area in Fig. 2). Parameters are $a = 0.9$, $k = 1$, $L = 1$, and $b = 1.5$.

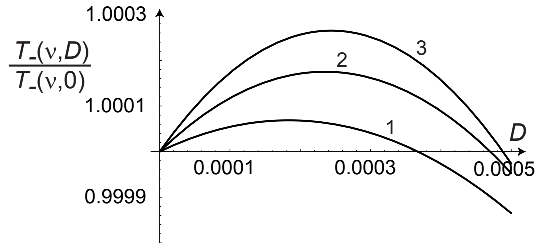

Fig. 4. The normalized MFPT vs the white noise intensity D, for three values of the dimensionless mean flipping rate $\omega = (\nu L)/k$: 0.7 (curve 1), 0.8 (curve 2), 0.9 (curve 3) (upper area in Fig. 2). Parameters are $a = 0.97$, $k = 1$, $L = 1$, and $b = 1.5$.

decreases, for fixed q, when ω decreases in the upper area of Figure 2, and when ω increases in the lower area of Figure 2. At the same time, the NES phenomenon for the upper area in Figure 2 is very small (see Fig. 4).

Now we analyse the behaviour of the escape times $T_-(\nu, D)$ and $T_+(\nu, D)$ as a function of the mean flipping rate ν. From equations (15) and (16) we obtain these behaviors, which are shown in Figure 5 for a fixed value of the white noise intensity $D = 0.1$.

We find the resonant activation phenomenon [26,28] for $T_+(\nu, D)$ and monotonic behaviour for $T_-(\nu, D)$. In the same Figure 5 we report the arithmetic average of these quantities: $\bar{T}(\nu, D) = [T_+(\nu, D) + T_-(\nu, D)]/2$, which is the average escape time analyzed by Doering and Gadoua in reference [26]. Starting from the metastable configuration, the mean first passage time $T_+(\nu, D)$ decreases with the mean flipping rate, because the potential fluctuations induce crossing events of the potential barrier. On the other hand, starting from the unstable configuration, the average escape time $T_-(\nu, D)$ increases monotonically because the potential fluctuations stabilize the initial unstable state. For very fast potential fluctuations, Brownian particles "forget" the initial configuration of the potential and all the mean first passage times ($T_+(\nu, D)$, $T_-(\nu, D)$, and $\bar{T}(\nu, D)$) join in the same asymptotic value: the average escape time required to cross the average barrier [26,28].

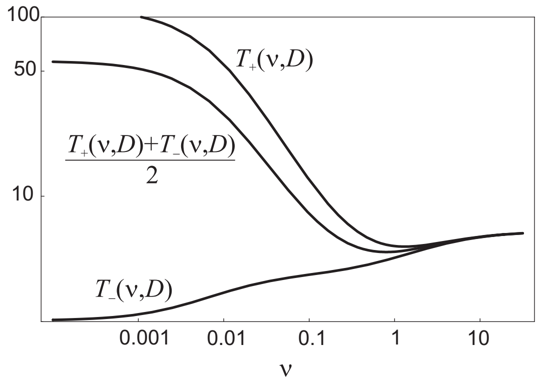

Fig. 5. Logarithmic plot of the mean first passage times: $T_+(\nu, D)$ (initial metastable configuration), $T_-(\nu, D)$ (initial unstable configuration), $\bar{T}(\nu, D)$ (arithmetic average), as a function the mean flipping rate ν. Here $D = 0.1$, $a = 0.5$ and the other parameters are the same as in Figures 3 and 4.

4 Conclusions

We have investigated the noise enhanced stability (NES) phenomenon in one-dimensional system with Gaussian additive white noise and a potential randomly switched by a Markovian dichotomous process. We have derived general equations to calculate the mean first passage times with one reflecting boundary and one absorbing boundary. For piece-wise linear potential we obtain the exact solution for MFPTs as a function of arbitrary white noise intensity and arbitrary mean flipping rate of the potential barrier. By analyzing the derived equations we obtain analytically the region of the NES phenomenon occurrence. We find that this effect can be observed only at very flattened sink beyond the potential barrier, i.e. at the real absence of the reverse potential barrier in the metastable state. Moreover we note that the noise enhanced stability phenomenon is related to rare events, that is when some particles are trapped in the metastable state. Therefore to better understand this effect it should be interesting to analyse the probability distribution of the escape time. This will be done in a forthcoming paper. By investigating the behaviour of the mean first passage times as a function of the mean flipping rate of the potential we find the resonant activation phenomenon for initial metastable configuration and monotonic behaviour for initial unstable configuration. With our theoretical approach we are able to describe, therefore, two different noise-induced effects. Our exact analytical results, concerning the enhancement of stability of metastable states for one-dimensional systems, can be a good starting point to extend and improve the same theoretical apparatus to more complex systems with fluctuating potential barriers.

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Appendix A

Let us consider a stochastic Langevin equation with two random forces

$$\dot{x} = f(x) + g(x)\,\eta(t) + h(x)\,\xi(t), \tag{25}$$

where $\xi(t)$ is the white Gaussian noise with zero mean and correlation function $\langle \xi(t)\,\xi(t+\tau)\rangle = 2D\delta(\tau)$, $\eta(t)$ is a Markovian dichotomous noise with flippings mean rate ν and values ± 1, $f(x)$, $g(x)$, and $h(x)$ are arbitrary functions. To derive the closed equation for the joint probability density of the random processes $x(t)$ and $\eta(t)$ we start from the following expression

$$W(x, y, t) = \langle \delta(x - x(t))\,\delta(y - \eta(t))\rangle \tag{26}$$

and then apply well-known method based on functional correlation formulae (see, for example, Refs. [35,36]).

Upon differentiation of equation (26) on t, we obtain

$$\frac{\partial W}{\partial t} = -\frac{\partial}{\partial x}\langle \dot{x}(t)\,\delta(x - x(t))\,\delta(y - \eta(t))\rangle + \left\langle \delta(x - x(t))\frac{\partial}{\partial t}\delta(y - \eta(t))\right\rangle. \tag{27}$$

Substituting $\dot{x}(t)$ from equation (25) and taking into account the definition (26), we can rewrite equation (27) as

$$\frac{\partial W}{\partial t} = -\frac{\partial}{\partial x}[f(x) + yg(x)]\,W - \frac{\partial}{\partial x}h(x)\langle \xi(t)\,\delta(x - x(t))\,\delta(y - \eta(t))\rangle + \left\langle \delta(x - x(t))\frac{\partial}{\partial t}\delta(y - \eta(t))\right\rangle. \tag{28}$$

To split the functional average in equation (28) we use Furutsu-Novikov's formula for the white Gaussian noise $\xi(t)$ [37,38]

$$\langle \xi(t)\,F_t[\xi]\rangle = D\left\langle \frac{\delta F_t[\xi]}{\delta\xi(t)}\right\rangle, \tag{29}$$

where $F_t[\xi]$ is an arbitrary functional depending on the history of random process $\xi(t)$ $(0 \le \tau \le t)$. Replacing $F_t[\xi]$ with $\delta(x - x(t))\,\delta(y - \eta(t))$ in equation (29) and taking into account that, in accordance with equation (25), $\delta x(t)/\delta\xi(t) = h(x(t))$, we obtain

$$\langle \xi(t)\,\delta(x - x(t))\,\delta(y - \eta(t))\rangle = -D\frac{\partial}{\partial x}h(x)\,W. \tag{30}$$

Further let us rearrange the last part of second average in equation (28). Using the translation operator and taking into account that for dichotomous noise with the values ± 1: $\eta^{2k}(t) = 1$, $\eta^{2k+1}(t) = \eta(t)$ $(k = 1, 2, \ldots)$, we arrive at

$$\frac{\partial}{\partial t}\delta(y - \eta(t)) = \frac{\partial}{\partial t}\exp\left(-\eta(t)\frac{d}{dy}\right)\delta(y) = -\frac{\partial}{\partial t}\sinh\left(\eta(t)\frac{d}{dy}\right)\delta(y) = -\dot{\eta}(t)\sinh\left(\frac{d}{dy}\right)\delta(y). \tag{31}$$

Substituting equations (30) and (31) in equation (28), we obtain

$$\frac{\partial W}{\partial t} = -\frac{\partial}{\partial x}\left[f(x) + yg(x)\right]W + D\frac{\partial}{\partial x}h(x)\frac{\partial}{\partial x}h\left(x\right)W - \left\langle \dot{\eta}\left(t\right)\delta(x - x(t))\sinh\left(\frac{d}{dy}\right)\delta(y)\right\rangle. \quad (32)$$

Now we use the Shapiro-Loginov's formula for a Markovian dichotomous noise $\eta(t)$ [39], to express the functional average in equation (32) in terms of the joint probability distribution

$$\left\langle \dot{\eta}\left(t\right)R_t\left[\eta\right]\right\rangle = -2\nu\left\langle \eta\left(t\right)R_t\left[\eta\right]\right\rangle, \quad (33)$$

where $R_t[\eta]$ is an arbitrary functional depending on the history of random process $\eta(\tau)$ $(0 \leq \tau \leq t)$. As a result, for the functional average in equation (32) we have

$$\left\langle \dot{\eta}\left(t\right)\delta(x - x(t))\sinh\left(\frac{d}{dy}\right)\delta(y)\right\rangle = -2\nu\left\langle \delta(x - x(t))\sinh\left(\eta\left(t\right)\frac{d}{dy}\right)\delta(y)\right\rangle = -\nu\left\langle \delta(x - x(t))\left[\delta(y + \eta(t)) - \delta(y - \eta(t))\right]\right\rangle. \quad (34)$$

Substituting equation (34) in equation (32) and taking into account equation (26) we obtain finally the following forward Kolmogorov's equation for the joint probability distribution

$$\frac{\partial W}{\partial t} = -\frac{\partial}{\partial x}\left[f\left(x\right) + yg\left(x\right)\right]W + D\left[\frac{\partial}{\partial x}h\left(x\right)\right]^2 W + \nu\left[W(x,-y,t) - W(x,y,t)\right]. \quad (35)$$

According to equation (35) we can reconstruct the backward Kolmogorov's equation for the conditional probability distribution $W(x,y,t\,|x_0,y_0,t_0\,)$ of the Markovian vector-process $\{x\left(t\right),\eta\left(t\right)\}$

$$-\frac{\partial W}{\partial t_0} = \left[f\left(x_0\right) + y_0 g\left(x_0\right)\right]\frac{\partial W}{\partial x_0} + D\left[h\left(x_0\right)\frac{\partial}{\partial x_0}\right]^2 W + \nu\left[W(x,y,t\,|x_0,-y_0,t_0\,) - W(x,y,t\,|x_0,y_0,t_0\,)\right]. \quad (36)$$

Since the conditional probability density depends only on the difference $\tau = t - t_0$, equation (36) can be rewritten as

$$\frac{\partial W}{\partial \tau} = \left[f\left(x_0\right) + y_0 g\left(x_0\right)\right]\frac{\partial W}{\partial x_0} + D\left[h\left(x_0\right)\frac{\partial}{\partial x_0}\right]^2 W + \nu\left[W(x,y,\tau\,|x_0,-y_0,0\,) - W(x,y,\tau\,|x_0,y_0,0\,)\right]. \quad (37)$$

Let us introduce the random first passage time θ. The probability $\Pr\{\theta > \tau\} \equiv P\left(x_0,y_0,\tau\right)$ that random process $x\left(t\right)$ remains between the absorbing boundaries $x = L_1$ and $x = L_2$ during the time interval $(0,\tau)$ is

$$P\left(x_0,y_0,\tau\right) = \int_{L_1}^{L_2} dx \int_{-\infty}^{\infty} W(x,y,\tau\,|x_0,y_0,0\,)dy. \quad (38)$$

Taking into account equation (38) and integrating equation (37) on x and y we find

$$\frac{\partial P}{\partial \tau} = \left[f\left(x_0\right) + y_0 g\left(x_0\right)\right]\frac{\partial P}{\partial x_0} + D\left[h\left(x_0\right)\frac{\partial}{\partial x_0}\right]^2 P + \nu\left[P(x_0,-y_0,\tau) - P(x_0,y_0,\tau)\right]. \quad (39)$$

To obtain the probability density of first passage time we should differentiate the probability $\Pr\{\theta > \tau\}$ on τ

$$w\left(x_0,y_0,\tau\right) = \frac{\partial \Pr\{\theta < \tau\}}{\partial \tau} = -\frac{\partial P(x_0,y_0,\tau)}{\partial \tau}.$$

As a result, we have the same equation (39) for $w\left(x_0,y_0,\tau\right)$

$$\frac{\partial w}{\partial \tau} = \left[f\left(x_0\right) + y_0 g\left(x_0\right)\right]\frac{\partial w}{\partial x_0} + D\left[h\left(x_0\right)\frac{\partial}{\partial x_0}\right]^2 w + \nu\left[w(x_0,-y_0,\tau) - w(x_0,y_0,\tau)\right]. \quad (40)$$

For the mean first passage time

$$\vartheta\left(x_0,y_0\right) = \int_0^{\infty} \tau w(x_0,y_0,\tau)d\tau \quad (41)$$

we obtain from equation (40) the following differential equation

$$D\left[h\left(x_0\right)\frac{\partial}{\partial x_0}\right]^2 \vartheta + \left[f\left(x_0\right) + y_0 g\left(x_0\right)\right]\frac{\partial \vartheta}{\partial x_0} + \nu\left[\vartheta(x_0,-y_0) - \vartheta(x_0,y_0)\right] = -1. \quad (42)$$

Since $y_0 = \pm 1$, equation (42) is equivalent to the following set of equations for times $T_+\left(x_0\right) \equiv \vartheta(x_0,+1)$ and $T_-\left(x_0\right) \equiv \vartheta(x_0,-1)$

$$\hat{L}_{x_0}T_+ + \left[f\left(x_0\right) + g\left(x_0\right)\right]\frac{dT_+}{dx_0} + \nu\left(T_- - T_+\right) = -1,$$
$$\hat{L}_{x_0}T_- + \left[f\left(x_0\right) - g\left(x_0\right)\right]\frac{dT_-}{dx_0} + \nu\left(T_+ - T_-\right) = -1, \quad (43)$$

where we introduced the operator

$$\hat{L}_{x_0} = D\left[h(x_0)\frac{d}{dx_0}\right]^2. \quad (44)$$

Equations (43) are similar to well-known equations for mean first passage times obtained in reference [33]. But authors used the Ito's interpretation of Langevin equation (25) and, as a result, obtained different equations. For $h\left(x\right) = 1$, equations (43) coincide with the equations derived in reference [33]. Finally, if we put in equations (43) and (44): $g\left(x\right) = -a$, $h\left(x\right) = 1$, we obtain equations (2).

Appendix B

We explain the origin of conditions (3) for the mean first passage times at a reflecting boundary. For sake of definiteness, we consider a left reflecting boundary at the point A. Let Brownian particle reaches the reflecting boundary at the instant $t_0 - \Delta t_0$. For the next small time period Δt_0, dichotomous noise $\eta(t)$ can switch to the opposite state with the probability $\beta(\Delta t_0)$ or remains in the initial state with the probability $(1 - \beta(\Delta t_0))$. The scenario of Brownian particle behaviour for this time interval is the following: it transfers to the point $A + \Delta x_0$ ($\Delta x_0 > 0$) with some fixed probability α or stays at reflecting boundary with the probability $(1 - \alpha)$. We denote the probability α as α_- for the case when dichotomous noise switches and as α_+ for the opposite case. As a result, the joint conditional probability distribution reads

$$W(x_0, z_0, t_0 \,|A, y_0, t_0 - \Delta t_0\,) = \beta(\Delta t_0)\delta\,(z_0 + y_0) \cdot [\alpha_- \delta\,(x_0 - A - \Delta x_0) + (1-\alpha_-)\,\delta\,(x_0 - A)] + [1 - \beta(\Delta t_0)]\,\delta\,(z_0 - y_0)\,[\alpha_+ \delta\,(x_0 - A - \Delta x_0) + (1-\alpha_+)\,\delta\,(x_0 - A)]. \quad (45)$$

Substituting equation (45) in Smoluchowski equation

$$W(x, y, t\,|A, y_0, t_0 - \Delta t_0\,) = \int_A^{+\infty} dx_0 \int_{-\infty}^{+\infty} dz_0 \, W(x_0, z_0, t_0\,|A, y_0, t_0 - \Delta t_0\,) W(x, y, t\,|x_0, z_0, t_0\,)$$

we obtain

$$W(x, y, t\,|A, y_0, t_0 - \Delta t_0\,) = \alpha_- \beta(\Delta t_0) W(x, y, t\,|A + \Delta x_0, -y_0, t_0\,) + \alpha_+ \left[1 - \beta(\Delta t_0)\right] W(x, y, t\,|A + \Delta x_0, y_0, t_0\,) + \beta(\Delta t_0)\,(1 - \alpha_-)\, W(x, y, t\,|A, -y_0, t_0\,) + (1 - \alpha_+)\left[1 - \beta(\Delta t_0)\right] W(x, y, t\,|A, y_0, t_0\,). \quad (46)$$

By expanding the conditional probability distributions in equation (46) in power series in Δt_0, up to a linear terms, and in Δx_0, up to a quadratic terms, we find approximately

$$-\Delta t_0 \frac{\partial W_+}{\partial t_0} \simeq \beta(\Delta t_0)\,(W_- - W_+) + \alpha_- \beta(\Delta t_0) \Delta x_0 \left(\frac{\partial W_-}{\partial x_0}\bigg|_A + \frac{\Delta x_0}{2} \frac{\partial^2 W_-}{\partial x_0^2}\bigg|_A \right) + \alpha_+ \Delta x_0 \left[1 - \beta(\Delta t_0)\right] \left(\frac{\partial W_+}{\partial x_0}\bigg|_A + \frac{\Delta x_0}{2} \frac{\partial^2 W_+}{\partial x_0^2}\bigg|_A \right), \quad (47)$$

where for simplicity we introduce the notations $W_+ = W(x, y, t\,|A, y_0, t_0\,)$ and $W_- = W(x, y, t\,|A, -y_0, t_0\,)$.

For Markovian dichotomous noise we have $\beta(\Delta t_0) = \nu \Delta t_0$. After substituting this value in equation (47) and dividing by $\alpha_+ \Delta x_0$ both sides of the equation, we arrive at

$$-\frac{\Delta t_0}{\alpha_+ \Delta x_0} \frac{\partial W_+}{\partial t_0} \simeq \frac{\nu \Delta t_0}{\alpha_+ \Delta x_0}\,(W_- - W_+) + (1 - \nu \Delta t_0) \left(\frac{\partial W_+}{\partial x_0}\bigg|_A + \frac{\Delta x_0}{2} \frac{\partial^2 W_+}{\partial x_0^2}\bigg|_A \right) + \nu \Delta t_0 \frac{\alpha_-}{\alpha_+} \left(\frac{\partial W_-}{\partial x_0}\bigg|_A + \frac{\Delta x_0}{2} \frac{\partial^2 W_-}{\partial x_0^2}\bigg|_A \right). \quad (48)$$

Since for a diffusion process $x(t)$: $\Delta x_0 \sim \sqrt{D \Delta t_0}$, from equation (48) we obtain in the limit $\Delta t_0 \to 0$

$$\frac{\partial W_+}{\partial x_0}\bigg|_A = 0. \quad (49)$$

By differentiating equation (41) (see Appendix A) and using equation (49) we find

$$T'_+\,(A) = 0, \quad T'_-\,(A) = 0, \quad (50)$$

that are the boundary conditions (3) used in our paper.

If we request an immediate switching of dichotomous noise $\eta(t)$ when Brownian particle reaches the reflecting boundary A, we must put $\beta(\Delta t_0) = 1$ in equation (47). As a result, we have

$$-\Delta t_0 \frac{\partial W_+}{\partial t_0} \simeq \alpha_- \Delta x_0 \left(\frac{\partial W_-}{\partial x_0}\bigg|_A + \frac{\Delta x_0}{2} \frac{\partial^2 W_-}{\partial x_0^2}\bigg|_A \right) + W_- - W_+. \quad (51)$$

In the limit $\Delta t_0 \to 0$ we obtain from equation (51)

$$W_+ = W_-. \quad (52)$$

Taking into account equality (52) in equation (51) and dividing both sides of this equation by $\alpha_- \Delta x_0$, we find in the limit $\Delta t_0 \to 0$

$$\frac{\partial W_-}{\partial x_0}\bigg|_A = 0. \quad (53)$$

Equations (52) and (53) are equivalent to the following conditions for MFPTs at the reflecting boundary

$$T_+\,(A) = T_-\,(A)\,, \quad T'_+\,(A) = 0 \quad (54)$$

and

$$T_+\,(A) = T_-\,(A)\,, \quad T'_-\,(A) = 0. \quad (55)$$

It would be emphasized that conditions (54) and (55) were previously derived in reference [33] by more complex procedure. We must choose the conditions (54) or, alternatively, the conditions (55) from physical considerations.

Appendix C

Using equations (16) we can eliminate the unknown constant c_6 from equation (15) and after rearrangements obtain

$$T_\pm\,(0) = \frac{b}{k} + \frac{\nu L^2}{\Gamma^2} + \sum_{i=1}^{3} a c_i \left(\frac{e^{\mu_i}}{k + \lambda_i D} + \frac{D \lambda_i}{\Gamma^2} \right) - \frac{\Gamma^2}{2\nu k^2} - c_5 \frac{2\nu L}{a} \mp c_5 \frac{\Gamma^2}{a^2} \left(1 \pm \frac{a}{k} \right). \quad (56)$$

The unknown constants c_1, c_2, c_3, c_5 involved in equation (56) must be found from the following set of equations (see Eqs. (16))

$$\sum_{i=1}^{3} c_i e^{\mu_i} = \frac{a}{2\nu k},$$
$$\sum_{i=1}^{3} c_i - c_5 \left(1 + \frac{2\nu D}{a^2}\cosh\gamma L\right)$$
$$= \frac{aD}{\Gamma^3}\sinh\gamma L + \frac{a}{2\nu k} - \frac{aL}{\Gamma^2},$$
$$\sum_{i=1}^{3} c_i\lambda_i - c_5\frac{2\nu\Gamma}{a^2}\sinh\gamma L = \frac{a}{\Gamma^2}(\cosh\gamma L - 1),$$
$$\sum_{i=1}^{3} \frac{c_i\lambda_i}{k+\lambda_i D} - c_5\frac{2\nu}{a^2}(\cosh\gamma L - 1)$$
$$= \frac{2\nu L}{a\Gamma^2} + \frac{a}{\Gamma^3}\sinh\gamma L - \frac{1}{ak}. \quad (57)$$

To investigate the NES phenomenon, we look for asymptotic expressions of equations (56) and (57) for $D \to 0$, with an accuracy of linear terms. To do this we need approximate expressions of the cubic equation roots. From equation (13) we have

$$\lambda_1 + \lambda_2 + \lambda_3 = -\frac{2k}{D},$$
$$\lambda_1\lambda_2\lambda_3 = \frac{2\nu k}{D^2}, \quad (58)$$

and, as a consequence, we can put the roots in the following form

$$\lambda \simeq \frac{A_1}{D} + A_2 + A_3 D, \quad (59)$$

with the unknown constants A_1, A_2, A_3. Substitution of equation (59) in equation (13) gives

$$\lambda_1 \simeq -\frac{k+a}{D} - \frac{\nu}{k+a},$$
$$\lambda_2 \simeq -\frac{k-a}{D} - \frac{\nu}{k-a},$$
$$\lambda_3 \simeq \frac{2\nu k}{k^2-a^2}\left[1 - \frac{2\nu D\left(k^2+a^2\right)}{\left(k^2-a^2\right)^2}\right]. \quad (60)$$

As it is seen from equation (60), in the limit $D \to 0$ ($\gamma \to +\infty$) we have: $\mu_1 \to -\infty$ and $\mu_2 \to -\infty$. Therefore, the negligibly small terms with e^{μ_1} and e^{μ_2} can be neglected in equation (56) and in the first equation (57), and we find

$$c_3 \simeq \frac{a}{2\nu k}\cdot e^{-\mu_3}. \quad (61)$$

Using the following approximate expressions

$$\sinh\gamma L \simeq \cosh\gamma L \simeq \frac{e^{\gamma L}}{2} \qquad (\gamma \to +\infty),$$

we find c_5 from equations (57)

$$c_5 \simeq -\frac{a^3}{2\nu\Gamma^3}. \quad (62)$$

Substituting equations (61) and (62) in equation (56), and solving the system (57), we obtain

$$T_\pm(0) \simeq \frac{b}{k} + \frac{\nu L^2}{\Gamma^2} + \frac{a^2}{2\nu k\,(k+\lambda_3 D)}$$
$$+\frac{a^2[k(\Gamma^2 - 4\nu D + 2\nu L\Gamma) - \Gamma^2(\Gamma + D\lambda_3)]}{\Gamma^4\lambda_3(k+D\lambda_3)(k+D\lambda_3+\Gamma)}$$
$$+\frac{a^2 e^{-\mu_3}[(k+D\lambda_3)(k+3D\lambda_3) - \Gamma^2]}{2\nu k\Gamma(k+D\lambda_3)(k+D\lambda_3+\Gamma)}$$
$$+\frac{a^2 L}{\Gamma^3} - \frac{\Gamma^2}{2\nu k^2} \pm \frac{a}{2\nu\Gamma}\left(1 \pm \frac{a}{k}\right). \quad (63)$$

After expanding the expression (63) in power series in D up to linear terms we derive the main result (17–19).

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